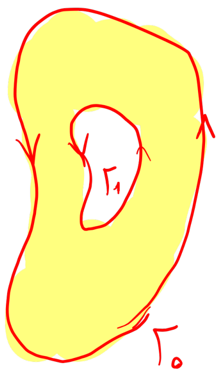
Analysis IV

(for EL, GM, MX)

Week 5 - 17.03.25

Recall from the previous weeks:

- As a consequence of Cauchy's theorem:



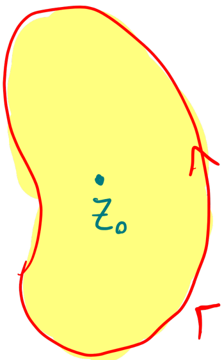
If Γ_0, Γ_1 are simple, closed curves oriented in the same direction, such that Γ_1 is inside Γ_0 and f is holomorphic between Γ_0 and Γ_1:

$$\int_{\Gamma_0} f(z)\, dz = \int_{\Gamma_1} f(z)\, dz$$

(In practice: I can move the loop of integration, as long as I don't "cross" any singularity of f).

- Cauchy's integral formula (extended):

If Γ is a simple, closed, counter-clockwise oriented curve and f is holomorphic in the interior of Γ, then



for any z_0 in the interior of Γ:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

- Taylor series: If $D \subseteq \mathbb{C}$ is open, $z_0 \in D$ and f is holomorphic in D: $\forall R>0$ such that the disk $B_R(z_0)$ lies entirely in D,

we have $$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

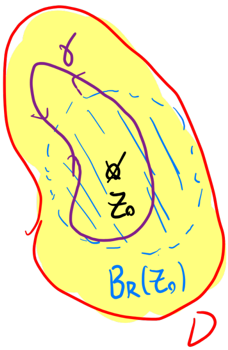
for any z in the disk $B_R(z_0)$.

The largest such R: "Radius of convergence" or "Radius of analyticity".

<u>In practice</u>: R is the distance from z_0 to the nearest "singular" point.

- Unlike in the real case for holomorphic functions we can also write down a series expansion even when f is not defined at z_0:

<u>Laurent series</u>: With D as before, if



$$f : D \setminus \{z_0\} \longrightarrow \mathbb{C} \text{ is holomorphic,}$$

then for any $R > 0$ such that $B_R(z_0) \subseteq D$, the following equality holds for $z \in B_R(z_0) \setminus \{z_0\}$:

(i.e. $z \neq z_0$!)

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n \cdot (z - z_0)^n ,$$

where the constant c_n can be computed by the formula:

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \quad 1 ,$$

where γ is any simple, closed, counter-clockwise curve around z_0 such that $\mathrm{int}(\gamma) \subseteq D$.

Remark: When f (originally defined on $D/\{z_0\}$) is actually extendible on z_0 as a holomorphic function there, then $C_n = 0$ for $n \leq -1$ and the Laurent series is simply the Taylor series. In this case, the fact that the coefficient C_n given by 1 vanishes for $n \leq -1$ follows by Cauchy's theorem (since $(z-z_0)^{-n-1}$ is holomorphic everywhere on $\mathbb{C}$ when $n \leq -1$).

- Laurent and Taylor series are unique: So if I can express $f(z)$ as a power series $\sum_{n=-\infty}^{+\infty} c_n (z-z_0)$ using some other method (not necessarily using 1), this is still the Laurent series.

Example

We can use the fact that $\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n$ for $|z|<1$ to expand:

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{+\infty} (-z^2)^n = \sum_{n=0}^{+\infty} (-1)^n z^{2n}$$

$$= \sum_{k=0}^{+\infty} c_k \cdot z^k$$

where $c_k = \begin{cases} 0, & k \text{ odd} \\ (-1)^{k/2}, & k \text{ even.} \end{cases}$

- Any Laurent series splits into its singular and regular parts:

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n \cdot (z-z_0)^n =$$

$$= \underbrace{\dots + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{z-z_0}}_{\text{singular part}} + \underbrace{c_0 + c_1(z-z_0) + \dots}_{\text{regular part}}$$

If f can be extended to z_0 so that it's holomorphic there: The singular part must vanish (and the regular part is just the Taylor series).

(in fact: The opposite is also true).

Example

Let $f: \mathbb{C}\setminus\{0,1\} \longrightarrow \mathbb{C}$, $\quad f(z) = \dfrac{1}{z\cdot(z+1)}$

What is the Laurent series of f at $z_0 \in \mathbb{C}$?

What is the radius of convergence?

We will consider cases:

$\boxed{\text{Case } z_0 = 0}$ Observe $f(z) = \frac{1}{z} - \frac{1}{z+1}$

Recall: $\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{+\infty} (-1)^n z^n$ for $|z|<1$.

So $f(z) = \frac{1}{z} - \sum_{n=0}^{+\infty} (-1)^n z^n = \sum_{n=-1}^{+\infty} (-1)^{n+1} z^n$.

Radius of convergence: $R = 1$
(equal to the distance of $z_0 = 0$ from the nearest point of "singularity" of f, in this case $z = 1$).

$\boxed{\text{Case } z_0 = -1}$ $f(z) = \frac{1}{z \cdot (z+1)} = -\frac{1}{z-(-1)} + \frac{1}{z}$

Expand $\frac{1}{z}$ in Taylor series around $z_0 = -1$:

$\frac{1}{z} = \sum_{n=0}^{+\infty} (-1) (z-(-1))^n$, converges for $|z-(-1)|<1$.

So $f(z) = -\frac{1}{z-(-1)} + \sum_{n=0}^{+\infty} (-1)(z-(-1))^n$

$$= \sum_{n=-1}^{+\infty} (-1)\,(z-(-1))^n \quad , \quad R = 1.$$

$\boxed{\text{Case } z_0 \neq 0, 1}$ In this case: f is holomorphic on a set containing z_0, so regular Taylor series. Converges in any disk centered at z_0 which does not contain 0 or -1.

So $R = \min\{|z_0|, |z_0+1|\}$.

Definitions:

- We say that f has a regular point at z_0 if the singular part of the Laurent series of f at z_0 vanishes.

Example: $f(z) = \sum_{n=0}^{+\infty} \frac{1}{n!} z^n \quad (= e^z)$

Remark: If f is holomorphic at z_0, then z_0 is also a regular point.

The converse is also true:

If z_0 is a regular point of f, then $\lim_{z \to z_0} f(z)$ exists and if we define f at z_0 so that $f(z_0) = \lim_{z \to z_0} f(z)$, then f is holomorphic at z_0.

Criterion that we will use:

z_0 is a regular point of f

$$\iff \lim_{z \to z_0} f(z) \text{ exists.}$$

- We say that f has a pole of order m at z_0 if the coefficients of the Laurent series of f at z_0 satisfy

$$c_n = 0 \quad \text{for } n < -m.$$

Example: $f(z) = \frac{1}{z}$ has a pole of order 1 at $z_0 = 0$.

$f(z) = \frac{1}{(z+2)^2} + \frac{1}{z}$ has a pole of order 2 at $z_0 = -2$ and a pole of order 1 at $z_0 = 0$.

In the case of a pole of order m at z_0:

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \dots + c_0 + c_1(z-z_0) + \dots$$

Note: $g(z) = (z-z_0)^m \cdot f(z)$ has a regular point at z_0!

- We say that f has an essential singularity at z_0 if the Laurent series has infinitely many coefficients $c_n \neq 0$ with $n \leq -1$.

Example: $f(z) = e^{1/z}$, $z_0 = 0$.

$$f(z) = \sum_{n=0}^{+\infty} \frac{(1/z)^n}{n!} = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n.$$

For $f(z) = z \cdot e^{1/z}$; $z_0 = 0$:

Laurent series:

$$z \cdot e^{1/z} = z \cdot \sum_{n=0}^{+\infty} \frac{(1/z)^n}{n!} = \underbrace{z + 1}_{\text{Regular part}} + \underbrace{\frac{1}{2! z} + \frac{1}{3! z^2} + \dots}_{\text{singular part}}$$

Suppose that $p, q : D \setminus \{z_0\} \to \mathbb{C}$ are holomorphic with z_0 being <u>regular</u> for both, and their Laurent series are given by:

$$p(z) = a_0 + a_1 \cdot (z - z_0) + a_2 \cdot (z - z_0)^2 + \dots$$

$$q(z) = b_0 + b_1 \cdot (z - z_0) + b_2 \cdot (z - z_0)^2 + \dots$$

(so p, q extend holomorphically at $z = z_0$)

We want to study the quotient $p(z)/q(z)$ near z_0. We want to know what is the order of the poles, if any.

We will first look at a few instructive examples.

a) If $b_0 \neq 0$ (so $q(z_0) \neq 0$):

$\frac{1}{q(z)}$ is holomorphic near z_0, and

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q(z_0)} = \frac{a_0}{b_0}$$

So z_0 is a regular point for f.

b) If $a_0 = b_0 = 0$ but $b_1 \neq 0$:

$$f(z) = \frac{a_1 (z-z_0) + a_2 \cdot (z-z_0)^2 + \dots}{b_1 \cdot (z-z_0) + b_2 \cdot (z-z_0)^2 + \dots} =$$

$$= \frac{a_1 + a_2 \cdot (z-z_0) + \dots}{b_1 + b_2 \cdot (z-z_0) + \dots}$$

So $\lim_{z \to z_0} f(z) = \frac{a_1}{b_1} \quad \left(= \frac{p'(z_0)}{q'(z_0)} \right)$

de L'Hôpital 's rule

So z_0 is a regular point for f.

Similarly if $a_0 = a_1 = 0$, $b_0 = b_1 = 0$ but $b_2 \neq 0$.

Example: Let $f(z) = \frac{z}{\sin(z)}$, $z_0 = 0$.

$$f(z) = \frac{z}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots} = \frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots}$$

So $\lim_{z \to 0} f(z) = 1$ Regular point ✓

Example: Let $f(z) = \frac{e^z - 1 - z}{z^2}$, $z_0 = 0$

$$e^z - 1 - z = \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

So

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{z^2/2! + z^3/3! + \dots}{z^2} = \frac{1}{2}$$

Regular point ✓

In Summary:

Either expand in power series and take out as many common powers of $(z-z_0)$ from the numerator and the denominator as possible and take the limit,

Or use de L'Hôpital's rule.

c) Suppose that

$$p(z) = a_k (z-z_0)^k + a_{k+1} (z-z_0)^{k+1} + \dots, \quad a_k \neq 0$$

and $$q(z) = b_l (z-z_0)^l + b_{l+1} (z-z_0)^{l+1} + \dots, \quad b_l \neq 0$$

and $$f(z) = p(z)/q(z)$$

- If $k \geqslant l$, then z_0 is a regular point:

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \frac{a_k (z-z_0)^k + a_{k+1} \cdot (z-z_0)^{k+1} + \dots}{b_l \cdot (z-z_0)^l + b_{l+1} \cdot (z-z_0)^{l+1} + \dots}$$

$$= \lim_{z \to z_0} \frac{a_k \cdot (z-z_0)^{k-l} + a_{k+1} \cdot (z-z_0)^{k-l+1} + \dots}{b_l + b_{l+1} (z-z_0) + \dots}$$

$$= \begin{cases} 0, & \text{if } k > l, \\ \frac{a_k}{b_l}, & \text{if } k = l. \end{cases}$$

The limit exists, so it's a regular point.

- If $k < l$: Then z_0 is a pole of order $l-k$:

$$f(z) = \frac{a_k \cdot (z-z_0)^k + a_{k+1} \cdot (z-z_0)^{k+1} + \dots}{b_l \cdot (z-z_0)^l + b_{l+1} \cdot (z-z_0)^{l+1} + \dots}$$

$$= \frac{a_k + a_{k+1}\cdot(z-z_0) + \dots}{b_l\cdot(z-z_0)^{l-k} + b_{l+1}\cdot(z-z_0)^{l-k+1} + \dots}$$

$$= \frac{1}{(z-z_0)^{l-k}} \cdot \underbrace{\frac{a_k + a_{k+1}\cdot(z-z_0) + \dots}{b_l + b_{l+1}\cdot(z-z_0) + \dots}}$$

For this: z_0 is a regular point, so its Laurent series has only a regular part: $= c_0 + c_1\cdot(z-z_0) + \dots$ where $c_0 = \frac{a_k}{b_l}$.

So $f(z) = \frac{c_0}{(z-z_0)^{l-k}} + \frac{c_1}{(z-z_0)^{l-k+1}} + \dots$

So top order term: $\frac{a_k/b_l}{(z-z_0)^{l-k}}$.

Example:

$f(z) = \dfrac{\sin(z)}{2z^3 + z^4}$, show it has a pole of order 2 at $z = 0$.

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$$

So $f(z) = \dfrac{z - z^3/3! + z^5/5! + \dots}{2z^3 + z^4} = \dfrac{1 - z^2/3! + z^4/5! + \dots}{2z^2 + z^3}$

$$= \frac{1}{z^2} \cdot \underbrace{\frac{1 - z^2/3! + \dots}{2 + z}}_{\text{Regular at } z_0 = 0}\,,$$

So near $z_0 = 0$: $f(z) = \dfrac{1}{2z^2} + \dfrac{c_{-1}}{z} + c_0 + \dots$